

The minimal number of generators for simple Lie superalgebras

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Abstract: Using the classification theorem due to Kac we prove that any finite dimensional simple Lie superalgebra over an algebraically closed field of characteristic 0 is generated by one element.

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0. Introduction

Throughout we work over an algebraically closed field of characteristic zero, \mathbb{F} , and all the vector spaces and algebras are assumed to be finite dimensional. Our principal aim is to determine the minimal number of generators for simple Lie superalgebras. We prove that any simple Lie superalgebra is generated by one element in the super sense, that is, any simple Lie superalgebra coincides with the smallest sub-Lie superalgebra containing some fixed element. We know that for finite simple groups only the groups of prime orders can be generated by one element and that a simple Lie algebra is never generated by one element. Our results are not surprising since in a Lie superalgebra the square of an odd element is even and not necessarily zero. As in finite simple group or simple Lie algebra cases, our proof is dependent on the classification theorem of simple Lie superalgebras [6], but not a one-by-one checking.

This study is mainly motivated by two papers of Bois mentioned as follows. In 2009, Bois [1] proved that any simple Lie algebra in arbitrary characteristic $p \neq 2, 3$ is generated by 2 elements and moreover, the classical Lie algebras and the graded Cartan type simple Lie algebras $W(1, \underline{n})$ (Zassenhaus algebras) can be generated by 1.5 elements, that is, any given nonzero element can be paired a suitable element such that these two elements generate the whole algebra. Later, as a continuation of this work, Bois [2] showed that the simple graded Lie algebras of Cartan type $W(m, \underline{n})$ with $m \neq 1$ and the ones of the remaining Cartan types S, H and K are never generated by 1.5 elements. Papers [1, 2] contain a considerable amount of

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information in characteristic p and cover the earlier results in characteristic 0: In 1976, Ionescu [5] proved that a simple Lie algebra over the field of complex numbers is generated by 1.5 elements; In 1951, Kuranashi [7] proved that a semi-simple Lie algebra in characteristic 0 is generated by 2 elements.

By the classification theorem [6], a simple Lie superalgebra (excluding simple Lie algebras) is either a classical Lie superalgebra or a Cartan Lie superalgebra (see also [8]). The Lie algebra (even part) of a classical Lie superalgebra is reductive and meanwhile there exists a similarity in the structure side between the Cartan Lie superalgebras in characteristic 0 and the simple graded Lie algebras of Cartan type in characteristic p . In view of the observation above, we began this study in 2009. Since the Lie algebra of a classical Lie superalgebra is reductive and the odd part decomposes into at most two irreducible components as adjoint modules (see [6] or [8]), we first proved that the Lie algebra of a classical Lie superalgebra is generated by two elements and then obtained that any classical Lie superalgebra is generated by two (non-homogeneous) elements in non-super sense; As expected, we also proved that any Cartan Lie superalgebra is generated by two (non-homogeneous) elements (see arXiv 1103. 4242v1 math. ph). Thus we obtained that any simple Lie superalgebra is generated by two (non-homogeneous) elements in the non-super sense and then we submitted the manuscript to a journal for publication. Later, a conversation with Professor Yucai Su during a workshop on Lie theory (organized by Professor Bin Shu, spring of 2011) led us to consider the question: Determine all the simple Lie superalgebras which are generated by one element??? (equivalently, by two homogeneous elements in the non-super sense).

Let us briefly explain the outline and ideas in this improved version. A simple fact is that $[L_{\bar{1}}, L_{\bar{1}}] = L_{\bar{0}}$ for a simple Lie superalgebra L . So, for a classical Lie superalgebra L , our discussion is mainly based on the weight decomposition of $L_{\bar{1}}$ as $L_{\bar{0}}$ -module relative to the standard Cartan subalgebra???: We find the desired generators by starting from the sum of all the odd weight vectors and use the fact that $L_{\bar{1}}$ as $L_{\bar{0}}$ -module is irreducible or a direct sum of two irreducible submodules. For a Cartan Lie superalgebra L , we know that L is generated by its local part $L_{-1} + L_0 + L_1$ with respect to the standard grading and moreover, the null L_0 is a Lie algebra and L_i as L_0 -module, $i = \pm 1$, is irreducible or a direct sum of two irreducible submodules. Then, considering the weight space decomposition relative to the standard Cartan subalgebra of L_0 ???, we find the desired generators by choosing a weight vector in each irreducible submodule of L_i . The proofs??? of main conclusions are constructive and provide an explicit description of the generator candidates. The process involves certain computational techniques and we use certain information about classical Lie superalgebras from [9].

In this paper we write $\langle X \rangle$ for the sub-Lie superalgebra generated by a subset X in a Lie superalgebra.

1. Classical Lie superalgebras

A classical Lie superalgebra by definition is a simple Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ for which $L_{\bar{1}}$ as $L_{\bar{0}}$ -module is completely reducible [6, 8]. The information of

classical Lie superalgebras is as follows [6]:

Table 1.1

L	$L_{\bar{0}}$	$L_{\bar{1}}$ as $L_{\bar{0}}$ -module
$A(m, n), m, n \geq 0, n \neq m$	$A_m \oplus A_n \oplus \mathbb{F}$	$\mathfrak{sl}_{m+1} \otimes \mathfrak{sl}_{n+1} \otimes \mathbb{F} \oplus (\text{its dual})$
$A(n, n), n > 0$	$A_n \oplus A_n$	$\mathfrak{sl}_{n+1} \otimes \mathfrak{sl}_{n+1} \oplus (\text{its dual})$
$B(m, n), m \geq 0, n > 0$	$B_m \oplus C_n$	$\mathfrak{so}_{2m+1} \otimes \mathfrak{sp}_{2n}$
$D(m, n), m \geq 2, n > 0$	$D_m \oplus C_n$	$\mathfrak{so}_{2m} \otimes \mathfrak{sp}_{2n}$
$C(n), n \geq 2$	$C_{n-1} \oplus \mathbb{F}$	$\mathfrak{osp}_{2n-2} \oplus (\text{its dual})$
$P(n), n \geq 2$	A_n	$\Lambda^2 \mathfrak{sl}_{n+1}^* \oplus S^2 \mathfrak{sl}_{n+1}$
$Q(n), n \geq 2$	A_n	$\text{ad} \mathfrak{sl}_{n+1}$
$D(2, 1; \alpha), \alpha \in \mathbb{F} \setminus \{-1, 0\}$	$A_1 \oplus A_1 \oplus A_1$	$\mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$
$G(3)$	$\mathfrak{G}_2 \oplus A_1$	$\mathfrak{G}_2 \otimes \mathfrak{sl}_2$
$F(4)$	$B_3 \oplus A_1$	$\mathfrak{spin}_7 \otimes \mathfrak{sl}_2$

From Table 1.1, one sees that $L_{\bar{0}}$ is reductive and $L_{\bar{1}}$ as $L_{\bar{0}}$ -module is irreducible or a direct sum of two irreducible submodules.

Throughout this section L denotes a classical Lie superalgebra with the standard Cartan subalgebra H . The corresponding weight (root) decompositions are

$$L_{\bar{0}} = H \oplus \bigoplus_{\alpha \in \Delta_{\bar{0}}} L_{\bar{0}}^{\alpha}, \quad L_{\bar{1}} = \bigoplus_{\beta \in \Delta_{\bar{1}}} L_{\bar{1}}^{\beta};$$

$$L = H \oplus \bigoplus_{\alpha \in \Delta_{\bar{0}}} L_{\bar{0}}^{\alpha} \oplus \bigoplus_{\beta \in \Delta_{\bar{1}}} L_{\bar{1}}^{\beta}.$$

Write

$$\Delta := \Delta_{\bar{0}} \cup \Delta_{\bar{1}} \quad \text{and} \quad L^{\gamma} := L_{\bar{0}}^{\gamma} \oplus L_{\bar{1}}^{\gamma} \quad \text{for } \gamma \in \Delta.$$

Note that the standard Cartan subalgebra of a classical Lie superalgebra is diagonal:

$$\text{adh}(x) = \gamma(h)x \quad \text{for } h \in H, x \in L^{\gamma}, \gamma \in \Delta.$$

Let V be a vector space and $\mathfrak{F} := \{f_1, \dots, f_n\}$ a finite set of non-zero linear functions on V . Write

$$\Omega_{\mathfrak{F}} := \{v \in V \mid \Pi_{1 \leq i \neq j \leq n} (f_i - f_j)(v) \neq 0\}.$$

It is a standard fact that $\Omega_{\mathfrak{F}} \neq \emptyset$ (see also [1, Lemma 2.2.1]). The following technical lemma is a basic fact in Linear Algebra. For convenience, we write down a proof:

Lemma 1.1. *Let \mathfrak{A} be an algebra. For $a \in \mathfrak{A}$ write L_a for the left-multiplication operator given by a . Suppose $x = x_1 + x_2 + \dots + x_n$ is a sum of eigenvectors of L_a associated with mutually distinct eigenvalues. Then all x_i 's lie in the subalgebra generated by a and x .*

Proof. Let λ_i be the eigenvalues of L_a corresponding to x_i . Suppose for a moment that all the λ_i 's are nonzero. Then

$$(L_a)^k(x) = \lambda_1^k x_1 + \lambda_2^k x_2 + \dots + \lambda_n^k x_n \quad \text{for } k \geq 1.$$

Our conclusion in this case follows from the fact that the Van der Monde determinate given by $\lambda_1, \lambda_2, \dots, \lambda_n$ is nonzero and thereby the general situation is clear. \square

Lemma 1.2. [8, Proposition 1, p.137]

- (1) If $L \neq A(1, 1)$, $P(3)$ or $Q(n)$ then $\dim L^\gamma = 1$ for every $\gamma \in \Delta$.
- (2) If $L \neq Q(n)$ then $0 \notin \Delta_{\bar{1}}$.

Notice that, for $L = A(1, 1)$, $P(3)$ or $Q(n)$, from Table 1.1 $L_{\bar{0}}$ is a semi-simple Lie algebra. Then by a standard result in Lie algebras (see [4]) we have

$$\dim L_{\bar{0}}^\alpha = 1 \quad \text{for } \alpha \in \Delta_{\bar{0}}, \quad H = \sum_{\alpha \in \Delta_{\bar{0}}} [L_{\bar{0}}^\alpha, L_{\bar{0}}^{-\alpha}]. \quad (1.1)$$

Theorem 1.3. Any classical Lie superalgebra is generated by 1 element.

Proof. By Lemma 1.2(1), we treat two cases separately:

Case 1. If $L \neq A(1, 1)$, $Q(n)$ or $P(3)$, then all the weight spaces are 1-dimensional. Choose any $h \in \Omega_{\Delta_{\bar{1}}} \subset H$ and an element $x = \sum_{\gamma \in \Delta_{\bar{1}}} x_{\bar{1}}^\gamma$, where $x_{\bar{1}}^\gamma$ is a weight vector of γ . By Lemmas 1.2(2) and 1.1, all components $x_{\bar{1}}^\gamma$ belong to $\langle x + h \rangle$. Since $\dim L^\gamma = 1$, we conclude that $L^\gamma \subset \langle x + h \rangle$ for $\gamma \in \Delta_{\bar{1}}$ and then $L_{\bar{1}} \subset L$. By [6, Proposition 1.2.7(1), p.20], $L_{\bar{0}} = [L_{\bar{1}}, L_{\bar{1}}]$ and then $\langle x + h \rangle = L$.

Case 2. Let $L = A(1, 1)$, $Q(n)$ or $P(3)$. In this case, there exists a weight space which is not 1-dimensional.

Let $L = A(1, 1)$. For simplicity, write e_{ij} for $e_{ij} + \mathbb{F}I_4$. By Table 1.1, let $L_{\bar{1}} = L_{\bar{1}}^1 \oplus L_{\bar{1}}^2$ be a direct sum of two irreducible $L_{\bar{0}}$ -modules. The standard basis of $A(1, 1)$ is listed below:

Table 1.2

$L_{\bar{0}}$	H	$e_{11} + e_{33}, e_{11} + e_{44}$	$L_{\bar{1}}$	$L_{\bar{1}}^1$	$e_{13}, e_{14}, e_{23}, e_{24}$
		$e_{12}, e_{21}, e_{34}, e_{43}$		$L_{\bar{1}}^2$	$e_{31}, e_{32}, e_{41}, e_{42}$

Let x be the sum of all the standard odd basis elements (weight vectors) in Table 1.2, that is,

$$x = e_{13} + e_{14} + e_{23} + e_{24} + e_{31} + e_{32} + e_{41} + e_{42}.$$

Choose an element $h \in \Omega_{\Delta_{\bar{1}}}$. Assert $\langle x + h \rangle = L$. To that aim, define ε_i^a to be the linear function on H given by $\varepsilon_i^a(e_{11} + e_{2+j, 2+j}) = \delta_{ij}$ for $1 \leq i, j \leq 2$. All the odd weights and the corresponding odd weight vectors are listed below:

Table 1.3

weights	ε_2^a	ε_1^a	$-\varepsilon_1^a$	$-\varepsilon_2^a$
vectors	e_{13}, e_{42}	e_{14}, e_{32}	e_{23}, e_{41}	e_{31}, e_{24}

Then, by Lemma 1.1 and Table 1.3, the elements

$$e_{13} + e_{42}, e_{14} + e_{32}, e_{23} + e_{41}, e_{31} + e_{24}$$

lie in $\langle x + h \rangle$. A direct computation shows that

$$\begin{aligned} e_{34} &= \frac{1}{2}[e_{14} + e_{32}, e_{24} + e_{31}], & e_{12} &= \frac{1}{2}[e_{14} + e_{32}, e_{13} + e_{42}], \\ e_{21} &= \frac{1}{2}[e_{24} + e_{31}, e_{23} + e_{41}], & e_{43} &= \frac{1}{2}[e_{13} + e_{42}, e_{23} + e_{41}]. \end{aligned}$$

Then by Table 1.2, all the standard even basis elements $e_{12}, e_{21}, e_{43}, e_{34}$ lie in $\langle x + h \rangle$. According to (1.1), we have $L_{\bar{0}} \subset \langle x + h \rangle$. As

$$[e_{14} + e_{32}, e_{21}] = -e_{24} + e_{31} \in \langle x + h \rangle,$$

we have $e_{24}, e_{31} \in \langle x + h \rangle$. Since $e_{24} \in L_{\bar{1}}^1$, $e_{31} \in L_{\bar{1}}^2$ (see Table 1.2) and $L_{\bar{1}}^i$ is irreducible as $L_{\bar{0}}$ -module, where $i = 1, 2$, we have $L_{\bar{1}} \subset \langle x + h \rangle$. Therefore $\langle x + h \rangle = L$.

Let $L = P(3)$. Note that $P(3)$ is a subalgebra of $A(3, 3)$ consisting of the matrices of the form: $\begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$, where $\text{tr}(a) = 0$, b is symmetric and c is skew-symmetric (see [6]). By Table 1.1, let $L_{\bar{1}} = L_{\bar{1}}^1 \oplus L_{\bar{1}}^2$ be a direct sum of two irreducible $L_{\bar{0}}$ -modules. The standard basis of $P(3)$ is as follows:

Table 1.4

$L_{\bar{0}}$	H	$e_{11} - e_{22} - e_{55} + e_{66}, \quad e_{11} - e_{33} - e_{55} + e_{77}, \quad e_{11} - e_{44} - e_{55} + e_{88}$
		$e_{12} - e_{65}, \quad e_{13} - e_{75}, \quad e_{14} - e_{85}, \quad e_{23} - e_{76}, \quad e_{24} - e_{86}, \quad e_{34} - e_{87},$ $e_{21} - e_{56}, \quad e_{31} - e_{57}, \quad e_{41} - e_{58}, \quad e_{42} - e_{68}, \quad e_{43} - e_{78}, \quad e_{32} - e_{67}$
$L_{\bar{1}}$	$L_{\bar{1}}^1$	$e_{15}, \quad e_{26}, \quad e_{18} + e_{45}, \quad e_{28} + e_{46}, \quad e_{38} + e_{47},$ $e_{37}, \quad e_{48}, \quad e_{16} + e_{25}, \quad e_{17} + e_{35}, \quad e_{27} + e_{36}$
	$L_{\bar{1}}^2$	$e_{52} - e_{61}, \quad e_{53} - e_{71}, \quad e_{54} - e_{81}, \quad e_{63} - e_{72}, \quad e_{64} - e_{82}, \quad e_{74} - e_{83}$

Let x be the sum of all standard odd basis elements (weight vectors) in Table 1.4. Choose an element $h \in \Omega_{\Delta_{\bar{1}}}$. Assert $\langle x + h \rangle = L$. To that aim, define ε_i^p to be the linear function on H given by

$$\varepsilon_i^p(e_{11} - e_{1+j, 1+j} - e_{55} + e_{5+j, 5+j}) = \delta_{ij}$$

for $1 \leq i, j \leq 3$. All the odd weights and the corresponding odd weight vectors are listed below:

Table 1.5

weight	$2\varepsilon_1^p + 2\varepsilon_2^p + 2\varepsilon_3^p$	$-2\varepsilon_1^p$	$-2\varepsilon_2^p$	$-2\varepsilon_3^p$
vectors	e_{15}	e_{26}	e_{37}	e_{48}
weight	$\varepsilon_1^p + \varepsilon_3^p$	$-\varepsilon_1^p - \varepsilon_2^p$	$-\varepsilon_1^p - \varepsilon_3^p$	
vectors	$e_{17} + e_{35}, e_{64} - e_{82}$	$e_{27} + e_{36}, e_{54} - e_{81}$	$e_{28} + e_{46}, e_{53} - e_{71}$	
weight	$\varepsilon_2^p + \varepsilon_3^p$	$\varepsilon_1^p + \varepsilon_2^p$	$-\varepsilon_2^p - \varepsilon_3^p$	
vectors	$e_{16} + e_{25}, e_{74} - e_{83}$	$e_{18} + e_{45}, e_{63} - e_{72}$	$e_{38} + e_{47}, e_{52} - e_{61}$	

By Lemma 1.1 and Table 1.5, one sees that $\langle x + h \rangle$ contains the following elements

$$e_{37}, \quad e_{48}, \quad e_{38} + e_{47} + e_{52} - e_{61}, \quad e_{16} + e_{25} + e_{74} - e_{83}, \quad e_{17} + e_{35} + e_{64} - e_{82},$$

$$e_{15}, \quad e_{26}, \quad e_{27} + e_{36} + e_{54} - e_{81}, \quad e_{18} + e_{45} + e_{63} - e_{72}, \quad e_{28} + e_{46} + e_{53} - e_{71}.$$

Lie superbrackets of the above odd elements yield

$$\begin{aligned} e_{57} - e_{31} &= [e_{37}, e_{28} + e_{46} + e_{53} - e_{71}], \quad e_{58} - e_{41} = [e_{48}, e_{27} + e_{36} + e_{54} - e_{81}], \\ e_{56} - e_{21} &= [e_{26}, e_{38} + e_{47} + e_{52} - e_{61}], \quad e_{12} - e_{65} = [e_{15}, e_{38} + e_{47} + e_{52} - e_{61}], \\ e_{13} - e_{75} &= [e_{15}, e_{28} + e_{46} + e_{53} - e_{71}], \quad e_{23} - e_{76} = [e_{26}, e_{18} + e_{45} + e_{63} - e_{72}], \\ e_{24} - e_{86} &= [e_{26}, e_{17} + e_{35} + e_{64} - e_{82}], \quad e_{14} - e_{85} = [e_{15}, e_{27} + e_{36} + e_{54} - e_{81}], \\ e_{34} - e_{87} &= [e_{37}, e_{16} + e_{25} + e_{74} - e_{83}], \quad e_{67} - e_{32} = [e_{37}, e_{18} + e_{45} + e_{63} - e_{72}], \\ e_{68} - e_{42} &= [e_{48}, e_{17} + e_{35} + e_{64} - e_{82}], \quad e_{78} - e_{43} = [e_{48}, e_{16} + e_{25} + e_{74} - e_{83}]. \end{aligned}$$

Then, according to Table 1.4 and (1.1) one sees $L_{\bar{0}} \subset \langle x + h \rangle$. Since

$$e_{18} + e_{45} - e_{63} + e_{72} = [e_{17} + e_{35} + e_{64} - e_{82}, e_{78} - e_{43}] \in \langle x + h \rangle,$$

and $e_{18} + e_{45} + e_{63} - e_{72} \in \langle x + h \rangle$, we have

$$e_{18} + e_{45}, e_{63} - e_{72} \in \langle x + h \rangle.$$

By Table 1.4 we have

$$e_{18} + e_{45} \in L_{\bar{1}}^1, e_{63} - e_{72} \in L_{\bar{1}}^2.$$

Then the irreducibility of $L_{\bar{1}}^i$ as $L_{\bar{0}}$ -module ensures that $L_{\bar{1}}^i \subset \langle x + h \rangle$, where $i = 1, 2$. Therefore, $\langle x + h \rangle = L$.

Let $L = Q(n)$. Note that $Q(n) = \tilde{Q}(n)/\mathbb{F}I_{2n+2}$ and $\tilde{Q}(n)$ is the subalgebra of $\mathfrak{sl}(n+1, n+1)$ consisting of the matrices of the form: $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $\text{tr}(b) = 0$ (see [6]). For simplicity, write e_{ij} for $e_{ij} + \mathbb{F}I_{2n+2}$. The standard basis of $Q(n)$ is listed below:

Table 1.6

$L_{\bar{0}}$	H	$e_{ii} + e_{n+1+i, n+1+i}, \quad 1 \leq i \leq n$
		$e_{ij} + e_{n+1+i, n+1+j}, \quad 1 \leq i \neq j \leq n+1$
$L_{\bar{1}}$		$e_{1, n+2} + e_{n+2, 1} - e_{i, n+1+i} - e_{n+1+i, i}, \quad 2 \leq i \leq n+1,$
		$e_{j, n+1+k} + e_{n+1+j, k}, \quad 1 \leq k \neq j \leq n+1$

Let x be the sum of all standard odd basis elements (weight vectors) in Table 1.6. Choose an element $h \in \Omega_{\Delta_{\bar{1}}}$. Assert $\langle x + h \rangle = L$. To that aim, define ε_i^q to be the linear function on H by $\varepsilon_i^q(e_{jj} + e_{n+1+j, n+1+j}) = \delta_{ij}$ for $1 \leq i, j \leq n$. All the odd weights and the corresponding odd weight vectors are as follows:

Table 1.7

weights	$\varepsilon_j^q - \varepsilon_k^q, \quad 1 \leq k \neq j \leq n$
vectors	$e_{j, n+1+k} + e_{n+1+j, k}, \quad 1 \leq k \neq j \leq n$
weights	$\varepsilon_j^q, \quad 1 \leq j \leq n$
vectors	$e_{j, 2n+2} + e_{n+1+j, n+1}, \quad 1 \leq j \leq n$
weights	$-\varepsilon_k^q, \quad 1 \leq k \leq n$
vectors	$e_{n+1, n+1+k} + e_{2n+2, k}, \quad 1 \leq k \leq n$
weights	0
vectors	$e_{1, n+2} + e_{n+2, 1} - e_{i, n+1+i} - e_{n+1+i, i}, \quad 2 \leq i \leq n+1$

By Lemma 1.1 and Table 1.7, one sees that $\langle x + h \rangle$ contains the following elements

$$\begin{aligned} & e_{j, n+1+k} + e_{n+1+j, k} \quad (1 \leq k \neq j \leq n), \\ & e_{j, 2n+2} + e_{n+1+j, n+1} \quad (1 \leq j \leq n), \\ & e_{n+1, n+1+k} + e_{2n+2, k} \quad (1 \leq k \leq n), \\ & \sum_{j=2}^{n+1} (e_{1, n+2} + e_{n+2, 1} - e_{j, n+1+j} - e_{n+1+j, j}). \end{aligned}$$

Write Z for $\sum_{j=2}^{n+1}(e_{1,n+2} + e_{n+2,1} - e_{j,n+1+j} - e_{n+1+j,j})$. Then

$$\begin{aligned} & [e_{i,n+1+k} + e_{n+1+i,k}, Z] \\ = & \delta_{ij}(e_{jk} + e_{n+1+j,n+1+k}) - \delta_{i1}(e_{1k} + e_{n+2,n+1+k}) \\ & + \delta_{k1}(e_{i1} + e_{n+1+i,n+2}) - \delta_{kj}(e_{ij} + e_{n+1+i,n+1+j}), \quad 1 \leq i \neq k \leq n+1. \end{aligned}$$

Hence

$$e_{ik} + e_{n+1+i,n+1+k} \in \langle x+h \rangle, \quad 1 \leq i \neq k \leq n+1.$$

So, by Table 1.6 and (1.1) we have $L_{\bar{0}} \subset \langle x+h \rangle$. Since x lies in $\langle x+h \rangle$ and $L_{\bar{1}}$ is irreducible as $L_{\bar{0}}$ -module (see Table 1.1), we have $L_{\bar{1}} \subset \langle x+h \rangle$. Furthermore, $\langle x+h \rangle = L$. □

2. Cartan Lie superalgebras

All the Cartan Lie superalgebras are listed below [6, 8]:

$W(n)$ with $n \geq 3$, $S(n)$ with $n \geq 4$, $\tilde{S}(2m)$ with $m \geq 2$ and $H(n)$ with $n \geq 5$.

Let $\Lambda(n)$ be the Grassmann superalgebra with generators ξ_1, \dots, ξ_n . For a k -shuffle $u := (i_1, i_2, \dots, i_k)$, that is, a strictly increasing sequence between 1 and n , we write $|u| := k$ and $x^u := \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k}$. Letting $\deg \xi_i = 1$, $i = 1, \dots, n$, we obtain the standard consistent \mathbb{Z} -grading of $\Lambda(n)$. Let us briefly describe the Cartan Lie superalgebras.

- $W(n) = \text{der} \Lambda(n)$ is \mathbb{Z} -graded, $W(n) = \bigoplus_{k=-1}^{n-1} W(n)_k$, where

$$W(n)_k = \text{span}_{\mathbb{F}} \{x^u \partial / \partial \xi_i \mid |u| = k+1, 1 \leq i \leq n\}.$$

- $S(n) = \bigoplus_{k=-1}^{n-2} S(n)_k$ is a \mathbb{Z} -graded subalgebra of $W(n)$, where

$$S(n)_k = \text{span}_{\mathbb{F}} \{D_{ij}(x^u) \mid |u| = k+2, 1 \leq i, j \leq n\}.$$

Hereafter, $D_{ij}(f) := \partial(f)/\partial \xi_i \partial / \partial \xi_j + \partial(f)/\partial \xi_j \partial / \partial \xi_i$ for $f \in \Lambda(n)$.

- $\tilde{S}(2m)$ is a subalgebra of $W(2m)$ and as a \mathbb{Z} -graded subspace,

$$\tilde{S}(2m) = \bigoplus_{k=-1}^{2m-2} \tilde{S}(2m)_k \quad \text{with } m \geq 2,$$

where

$$\begin{aligned} \tilde{S}(2m)_{-1} &= \text{span}_{\mathbb{F}} \{(1 + \xi_1 \cdots \xi_{2m}) \partial / \partial \xi_j \mid 1 \leq j \leq 2m\}, \\ \tilde{S}(2m)_k &= S(2m)_k, \quad 0 \leq k \leq 2m-2. \end{aligned}$$

Notice that $\tilde{S}(2m)$ is not a \mathbb{Z} -graded subalgebra of $W(2m)$.

- $H(n) = \bigoplus_{k=-1}^{n-3} H(n)_k$ is a \mathbb{Z} -graded subalgebra of $W(n)$, where

$$H(n)_k = \text{span}_{\mathbb{F}} \{D_H(x^u) \mid |u| = k+2\}.$$

Hereafter, D_H is a linear mapping of $\Lambda(n)$ to $W(n)$ such that $D_H(f) := (-1)^{|f|} \sum_{i=1}^n \partial(f)/\partial \xi_i \partial / \partial \xi_{i'}$ for $f \in \Lambda(n)$, where $'$ is the involution of the index set $\{1, \dots, n\}$ satisfying that $i' = i + [\frac{n}{2}]$ for $i \leq [\frac{n}{2}]$ and $n' = n$ if n is odd. Here, $[\frac{n}{2}]$ is the biggest integer less than $\frac{n}{2}$ ($n \geq 5$).

In the sequel, we write W, S, \tilde{S}, H instead of $W(n), S(n), \tilde{S}(2m), H(n)$, respectively. Throughout this section L denotes one of the Cartan Lie superalgebras W, S, \tilde{S} , or H . Consider its decomposition of subspaces:

$$L = L_{-1} \oplus \cdots \oplus L_s. \quad (2.1)$$

For W, S, \tilde{S} or H , the height $s = n-1, n-2, 2m-2$ or $n-3$, respectively. Note that S and H are \mathbb{Z} -graded subalgebras of W with respect to (2.1), but \tilde{S} is not. The null L_0 is isomorphic to $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{sl}(2m), \mathfrak{so}(n)$ for $L = W, S, \tilde{S}, H$, respectively.

From [6, 8], we can write down the following facts:

Lemma 2.1. *Keep notations as above.*

- (1) *The subspace L_{-1} is an irreducible L_0 -module.*
- (2) *A Cartan Lie superalgebra L is generated by the local part $L_{-1} \oplus L_0 \oplus L_1$.*
- (3) *The subspace L_1 is an irreducible L_0 -module for $L = S, \tilde{S}$ or H , except for $H(6)$. For $L = H(6)$ or W , the subspace L_1 is a direct sum of two irreducible L_0 -submodules.*

The following is a list of bases of the standard Cartan subalgebras \mathfrak{h}_{L_0} of L_0 .

Table 2.1

L	\mathfrak{h}_{L_0}
$W(n)$	$\xi_i \partial / \partial \xi_i, 1 \leq i \leq n$
$S(n)$	$\xi_1 \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_j, 2 \leq j \leq n$
$\tilde{S}(2m)$	$\xi_1 \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_j, 2 \leq j \leq 2m$
$H(2m)$	$\xi_i \partial / \partial \xi_i - \xi_{i'} \partial / \partial \xi_{i'}, 1 \leq i \leq m$
$H(2m+1)$	$\xi_i \partial / \partial \xi_i - \xi_{i'} \partial / \partial \xi_{i'}, 1 \leq i \leq m$

The weight space decomposition of the subspace L_k relative to \mathfrak{h}_{L_0} is:

$$L_k = \delta_{k,0} \mathfrak{h}_{L_0} \oplus_{\alpha \in \Delta_k} L_k^\alpha, \text{ where } -1 \leq k \leq s.$$

We write down the following weight sets which will be used in the proof of the following Lemma 2.2. For $W(n)$, define ε_i^w to be the linear function on \mathfrak{h}_{W_0} by

$$\varepsilon_i^w(\xi_j \partial / \partial \xi_j) = \delta_{ij}, 1 \leq i, j \leq n.$$

We have

$$\begin{aligned} \Delta_{-1} &= \{-\varepsilon_j^w \mid 1 \leq j \leq n\}, \\ \Delta_1 &= \{\varepsilon_k^w + \varepsilon_l^w - \varepsilon_j^w \mid 1 \leq k \neq l, j \leq n\}. \end{aligned} \quad (2.2)$$

For $S(n)$ and $\tilde{S}(n)$, define ε_i^s to be the linear function on \mathfrak{h}_{S_0} by

$$\varepsilon_i^s(\xi_1 \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_j) = \delta_{ij}, 2 \leq i, j \leq n$$

and write $\varepsilon_1^s := \sum_{l=2}^n \varepsilon_l^s$. We have

$$\begin{aligned} \Delta_{-1} &= \{\varepsilon_j^s \mid 1 \leq j \leq n\}, \\ \Delta_1 &= \{\varepsilon_k^s + \varepsilon_l^s - \varepsilon_j^s \mid 1 \leq k \neq l, j \leq n\}. \end{aligned}$$

For $H(2m)$, define ε_i^h to be the linear function on \mathfrak{h}_{H_0} by

$$\varepsilon_i^h(\xi_j \partial / \partial \xi_j - \xi_{j'} \partial / \partial \xi_{j'}) = \delta_{ij}, \quad 1 \leq i, j \leq m.$$

We have

$$\begin{aligned} \Delta_{-1} &= \{ \pm \varepsilon_j^h \mid 1 \leq j \leq m \}, \\ \Delta_1 &= \{ \pm(\varepsilon_i^h + \varepsilon_j^h) \pm \varepsilon_k^h, \pm(\varepsilon_i^h - \varepsilon_j^h) \pm \varepsilon_k^h \mid 1 \leq i < j < k \leq m \} \\ &\quad \cup \{ \pm \varepsilon_l^h \mid 1 \leq l \leq m \}. \end{aligned} \quad (2.3)$$

For $H(2m+1)$, define $\varepsilon_i^{h'}$ to be the linear function on \mathfrak{h}_{H_0} by

$$\varepsilon_i^{h'}(\xi_j \partial / \partial \xi_j - \xi_{j'} \partial / \partial \xi_{j'}) = \delta_{ij}, \quad 1 \leq i, j \leq m.$$

We have

$$\begin{aligned} \Delta_{-1} &= \{0\} \cup \{ \pm \varepsilon_i^{h'} \mid 1 \leq i \leq m \}, \\ \Delta_1 &= \{0\} \cup \{ \pm \varepsilon_l^{h'}, \pm(\varepsilon_i^{h'} + \varepsilon_j^{h'}), \pm(\varepsilon_i^{h'} - \varepsilon_j^{h'}) \mid 1 \leq l \leq m, 1 \leq i < j \leq m \} \\ &\quad \cup \{ \pm(\varepsilon_i^{h'} + \varepsilon_j^{h'}) \pm \varepsilon_k^{h'}, \pm(\varepsilon_i^{h'} - \varepsilon_j^{h'}) \pm \varepsilon_k^{h'} \mid 1 \leq i < j < k \leq m \}. \end{aligned}$$

For $L_1 = H(6)_1$ or W_1 , then by Lemma 2.1(3), L_1 is a direct sum of two irreducible L_0 -modules

$$L_1 = L_1^1 \oplus L_1^2.$$

Let Δ_1^i be the weight set of L_1^i , $i = 1, 2$.

Lemma 2.2. *With the above nonations, we have the following properties:*

- (1) *If $L = W$ then $\Delta_{-1} \cap \Delta_1 = \emptyset$.*
- (2) *If $L = S$ or \tilde{S} then $\Delta_{-1} \cap \Delta_1 = \emptyset$.*
- (3) *If $L = H$ then $\Delta_{-1} \neq \Delta_1$.*
- (4) *If $L = H(6)$ or W , there exist nonzero weights $\alpha_1^i \in \Delta_1^i$ such that $\alpha_1^1 \neq \alpha_1^2$.*

Proof. All the statements follow directly from the above computations except (4) for $L = H(6)$ or W . In this case, from (2.2) and (2.3) one sees that $0 \notin \Delta_1$ and $|\Delta_1| > 1$. Consequently, (4) holds. \square

For $L = W, S, \tilde{S}$ or H , fix the corresponding standard Cartan subalgebra, respectively.

Lemma 2.3. *For L , there exists a weight vector $x_1 \in L_1^\alpha$ for some $\alpha \in \Delta_1$ such that $[x_1, x_1] = 0$.*

Proof. It is easy to see that a standard basis vector of L_1 is also a weight vector for some weight $\alpha \in \Delta_1$. For $L = W$, we have

$$[x_j x_k \partial / \partial \xi_i, x_j x_k \partial / \partial \xi_i] = 0$$

where $1 \leq i, j, k \leq n$ and i, j, k are pairwise distinct. For $L = S$ or \tilde{S} , we have

$$[D_{ij}(x_i x_j x_k), D_{ij}(x_i x_j x_k)] = 0$$

where $1 \leq i, j, k \leq n$ and i, j, k are pairwise distinct. For $L = H$, we have

$$[D_H(x_i x_j x_k), D_H(x_i x_j x_k)] = 0$$

where $1 \leq i, j, k \leq \lfloor \frac{n}{2} \rfloor$ and i, j, k are pairwise distinct. \square

Recall that the null L_0 is isomorphic to $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{sl}(2m), \mathfrak{so}(n)$ for $L = W, S, \tilde{S}$ or H , respectively.

Let \mathfrak{g} be a simple Lie algebra. If \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} , $x \in \mathfrak{g}$ is called \mathfrak{h} -balanced provided that $x^\alpha \neq 0$ for $\alpha \in \Phi$, where $\Phi \subset \mathfrak{h}^*$ is the root system of \mathfrak{g} relative to \mathfrak{h} . The lemma below is abstracted from [1]:

Lemma 2.4. *If x is a non-zero element in \mathfrak{g} then there exists some Cartan subalgebra \mathfrak{h} such that x is \mathfrak{h} -balanced.*

Proof. Suppose x is a non-zero element of \mathfrak{g} and let \mathfrak{h}' be a Cartan subalgebra of \mathfrak{g} . By the proof of [1, Theorem 2.2.3], there exists $\varphi \in \text{Aut } \mathfrak{g}$ such that $\varphi(x)$ is \mathfrak{h}' -balanced. Letting $\mathfrak{h} = \varphi^{-1}(\mathfrak{h}')$, one sees that \mathfrak{h} is a Cartan subalgebra and x is \mathfrak{h} -balanced. \square

From [1], we write down two useful facts:

- If $x \in \mathfrak{g}$ is \mathfrak{h} -balanced, then $\mathfrak{g} = \langle x, h \rangle$ for $h \in \Omega_\Phi$, where $\Phi \subset \mathfrak{h}^*$ is the root system of \mathfrak{g} relative to \mathfrak{h} .
- If $\mathfrak{sl}(n)$ is generated by an \mathfrak{h} -balanced element x and an element h in Ω_Φ , then $\mathfrak{gl}(n)$ is generated by x and $h + z$, where z is a nonzero central element in $\mathfrak{gl}(n)$.

Note that the \mathbb{Z} -grading of a Cartan Lie superalgebra is consistent with the \mathbb{Z}_2 -grading over \mathbb{F} .

Theorem 2.5. *Any Cartan Lie superalgebra is generated by 1 element.*

Proof. For $L = S, \tilde{S}$ or H , except $H(6)$, fix the standard Cartan subalgebra \mathfrak{h} . By Lemmas 2.2 and 2.3 we choose $\alpha_{-1} \neq \alpha_1$ for $\alpha_i \in \Delta_i$, $i = -1, 1$ and $x_{-1} \in L_{-1}^{\alpha_{-1}-1}$ and $x_1 \in L_1^{\alpha_1}$ such that $[x_{-1}, x_1] \neq 0$ and $[x_1, x_1] = 0$. Let $x_0 := 2[x_{-1}, x_1]$. Choose a suitable Cartan subalgebra \mathfrak{h}' such that x_0 is an \mathfrak{h}' -balanced element of L_0 . Let

$$x := x_{-1} + x_0 + h' + x_1$$

for $h' \in \Omega_{\Delta'_0}$, where $\Delta'_0 \subset \mathfrak{h}'^*$ is the root system of L relative to \mathfrak{h}' . Then we have

$$x_{-1} + x_1, x_0 + h' \in \langle x \rangle$$

and then

$$x_0 = [x_{-1} + x_1, x_{-1} + x_1] = 2[x_{-1}, x_1] \in \langle x \rangle.$$

Furthermore, we have $h' \in \langle x \rangle$. Then $L_0 = \langle x_0, h' \rangle \subset \langle x \rangle$. Choose an element

$$h \in \Omega_{\{\alpha_{-1}, \alpha_1\}} \subset \mathfrak{h} \subset L_0.$$

Then, by Lemma 1.1 we obtain that x_{-1} and x_1 lie in $\langle x \rangle$. According to Lemma 2.1(1) and (3), the irreducibility of L_{-1} and L_1 as L_0 -modules ensures that $L_i \subset \langle x \rangle$, $i = -1, 1$. Furthermore, $L = \langle x \rangle$.

For $L = H(6)$ or W , fix the standard Cartan subalgebra \mathfrak{h} . From Lemmas 2.2 and 2.3 we choose α_{-1}, α_1^1 and α_1^2 are pairwise distinct for $\alpha_{-1} \in \Delta_{-1}$, $\alpha_1^i \in \Delta_1^i$, $i = 1, 2$

and $x_{-1} \in L_{-1}^{\alpha_{-1}}$ and $x_1^i \in L_1^{\alpha_1^i}$ for $i = 1, 2$ such that $[x_{-1}, x_1^1 + x_1^2] \neq 0$ and $[x_1^1 + x_1^2, x_1^1 + x_1^2] = 0$. Let $x_0 := 2[x_{-1}, x_1^1 + x_1^2]$. Choose a suitable Cartan subalgebra \mathfrak{h}' such that x_0 is an \mathfrak{h}' -balanced element of L_0 . Let

$$x := x_{-1} + x_0 + \delta_{L,W}z + h' + x_1^1 + x_1^2$$

for $0 \neq z \in C(W_0)$ and $h' \in \Omega_{\Delta'_0}$, where $\Delta'_0 \subset \mathfrak{h}'^*$ is the root system of L relative to \mathfrak{h}' . Then we have

$$x_{-1} + x_1^1 + x_1^2, \quad x_0 + \delta_{L,W}z + h' \in \langle x \rangle$$

and then

$$x_0 = [x_{-1} + x_1^1 + x_1^2, x_{-1} + x_1^1 + x_1^2] = 2[x_{-1}, x_1^1 + x_1^2] \in \langle x \rangle.$$

Furthermore, we have $h' + \delta_{L,W}z \in \langle x \rangle$. Then

$$L_0 = \langle x_0, h' + \delta_{L,W}z \rangle \subset \langle x \rangle.$$

Choose an element

$$h \in \Omega_{\{\alpha_{-1}, \alpha_1^1, \alpha_1^2\}} \subset \mathfrak{h} \subset L_0.$$

Then, by Lemma 1.1 we obtain that x_{-1}, x_1^1 and x_1^2 lie in $\langle x \rangle$. According to Lemma 2.1(1) and (3), the irreducibility of L_{-1} , L_1^1 and L_1^2 as L_0 -modules ensures that $L = \langle x \rangle$. \square

Theorems 1.3 and 2.5 combine to the main result of this paper:

Theorem 2.6. *Any simple Lie superalgebra is generated by 1 element.*

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